Indian Statistical Institute
Second Semester Back Paper Exam 2005-2006
B.Math. (Hons.) I Year
Analysis II

Time: 3 hrs Date: -06-06

1. a) Let $f : [0,1] \longrightarrow R$ be any continuous function. Show that f is Riemann integrable. [5]

   b) Also show that

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f\left(\frac{j}{n}\right)$$

[5]

2. Let $f : R^2 \longrightarrow R$ have total derivative the linear map $A : R^2 \to R$ at $\underset{\sim}{0} = (0,0)$. Show that

   a) $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial y}(0,0)$ exist. [2]

   b) $\lim\limits_{t\to 0} \frac{f(t\underset{\sim}{u}) - f(\underset{\sim}{0})}{t}$ exists for any vector and is $A\underset{\sim}{u}$. [2]

   c) $A : R^2 \longrightarrow R$ satisfies $|A\underset{\sim}{u}| \le \|A\|_2 \|\underset{\sim}{u}\|_2$ [1]

   d) Show that f is continuous at $\underset{\sim}{0}$. $a, b \ge 0$ [2]

3. a) Let $0 \le c \le a+b$. Show that $\frac{c}{1+c} \le \frac{a}{1+a} + \frac{b}{1+b}$. [4]

   b) If (X,d) is a metric space, show that (X,m) is also a metric space where $m = \frac{d}{1+d}$ [2]

   c) Let X, d, m as in (b) show that $a_n \to a$ in $(X,d) \Leftrightarrow a_n \to a$ in (X,m) [3]

   d) Show that (X,d) and (X,m) have the same family of open sets. [3]

4. Let $(X_1, d_1), (X_2, d_2), (X_1 \times X_2, u)$ be metric spaces where

$$u((x_1,x_2),(a_1,a_2)) = \sqrt{[d_1(x_1,a_1)]^2 + [d_2(x_2,a_2)]^2}$$

   a) $\{(x_n, a_n)\}$ is Cauchy in $(X_1 \times X_2, u) \Leftrightarrow$ $\{x_n\}$ is Cauchy in (X_1, d_1) and $\{a_n\}$ is Cauchy seq in (X_2, d_2) [3]

   b) $(X_1 \times X_2, u)$ is complete $\Leftrightarrow$ (X_1, d_1) is complete and (X_2, d_2) is complete. [3]

5. Let A_1, A_2 be connected subsets of

   a) (X, d) with $A_1 \cap A_2 \neq$ empty. Show that $A_1 \cup A_2$ is connected. [4]

   b) Show that $\{(x, y) : x^2 + y^2 = 1\} \cup [1, \infty) \times \{0\}$ of R^2 is a connected set. [3]

6. a) Let (X, d) be compact. Show that (X, d) has a countable dense set. [4]

   b) Let (X, d) be compact. Show that d is bounded. [2]

7. Show that $[0, 1] \times [0, 1]$ is a compact subset of R^2. [7]

8. Let (X, d) be compact. Then every closed subset of (X, d) is also compact. [3]

9. Show that $f : (0, 1) \to R$ given by $f(x) = \sin(\frac{1}{x})$ is not uniformly continuous. [2]